of r, $m < n$) in several numerical examples. This should not be to surprising as Moore[6] has shown in the state space that, if the rank $B = r$, which it does here, the matrix GC is unique for a given set of eigenvalues. The gain matrix of the inverse method given in Eq. (26) and that of the standard approach of Eq. (15) can also be derived in the state-space-context as given in Refs. 1 and 7. Repeated numerical examples using MATLAB also seem to indicate that the gain matrix computed using Eq. (26) is computed in about 1/3 the time (about 0.134 s compared to 0.43 s for 4 DOF). This is largely due to calculating T^{-1} in the approach of Ref. 1. MATLAB codes for both gains are available from the author,[7] as are examples of low-order systems.

Conclusion

An eigenstructure assignment method has been presented based on using inverse eigenvalue theory. The resulting gain matrix does not resemble that obtained by conventional eigenstructure assignment but appears to produce the same numerical values for several different example problems, for less computational effort, when programmed in MATLAB. In addition, the proposed method allows both the open-loop structure and the closed-loop system to have rigid-body modes or very low frequencies often useful in aerospace structures. The remaining modes are stable because of the partitioning offered by the inverse eigenvalue approach when used with full state feedback. Often, full state feedback is looked upon with destain as being impractical. However, recent developments in smart structures technology have rendered full state feedback feasable.[8] The method proposed here allows specific eigenvalues to be changed, leaving the others unchanged.

The condition provided in Eq. (16) for physical mode shapes of mechanical systems should provide insight for those interested in control system design of mechanical systems where the assigned eigenvector is directly related to the physical vibration mode shape. Reference 1 should be consulted for an excellent physical example of flight control in state space. The result presented here makes a strong connection between the seemingly independent fields of inverse eigenvalue problems and that of eigenstructure assignment. Perhaps others can make more fruitful connections.

Acknowledgments

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References

[1]Andry, A. N., Shapiro, E. Y., and Chung, J. C., "Eigenstructure Assignment for Linear Systems," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 19, No. 5, 1983, pp. 711–729.

[2]Andry, A. N., Shapiro, E. Y., and Chung, J. C., "Modal Control for a Class of Linear, Discrete, Vibrating Mechanical Systems," *Proceedings 1st International Modal Analysis Conference* (Schenectady, NY), 1982, pp. 430–435.

[3]Starek, L., and Inman, D. J., "On the Inverse Vibration Problem with Rigid-Body Modes," *Journal of Applied Mechanics*, Vol. 50, Dec. 1991, pp. 1101–1104.

[4]Juang, J.-N., and Maghami, P. G., "Robust Eigenstructure Assignment for Second Order Dynamical Systems," *Mechanics and Control of Large Space Structures*, edited by J. L. Junkins, Progress in Astronautics and Aeronautics, Vol. 129, AIAA, Washington, DC, 1990, pp. 373–388.

[5]Juang, J.-N., Lim, K., Junkins, J. L., "Robust Eigenstructure Assignment for Flexible Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 3, 1989, pp. 381–387.

[6]Moore, B. C., "On the Flexibility Offered by State Feedback in Multivariable System Beyond Closed Loop Eigenvalue Assignment," *IEEE Transactions on Automatic Control*, Vol. A 1-21, 1976, pp. 689–692.

[7]Kress, A., "Model Correction and Control by Inverse Problems," Masters Thesis, Dept. of Mechanical and Aerospace Engineering, State Univ. of New York at Buffalo, NY, Oct. 1991.

[8]Leo, D. J., and Inman, D. J., "Pointing Control and Vibration Suppression of a Slewing Flexible Frame," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 3, 1994, pp. 329–536.

Variance-Based Sensor Placement for Modal Identification of Structures

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Introduction

In planning a structural identification test one is faced with a problem of where to place a limited number of sensors to maximize the accuracy of the experiments. Such considerations become more important when the sensors cannot be easily moved as in large space structures or when they are to be used for health monitoring.

Various methods for sensor and actuator positioning for control schemes have been developed, and examples are given in Refs. 1 and 2. In particular, Ref. 1 uses a backward elimination approach to pick locations for a linear quadratic regulator (LQR) control scheme. Methods for sensor placement in structural identification are contained in Refs. 3–5. Also within the structural identification field, Kammer[6] formulated and evaluated an effective independence (EfI) approach for the ranking of sensors which is also a backward elimination approach. The motivation was to pick sensor locations to render the resulting columns of the modal matrix as independent as possible. This was accomplished indirectly by considering the solution of a linear equation in which the modal matrix is the coefficient matrix. The accuracy of the resulting solution, measured by the size of the covariance matrix, is related to the linear independence of the modal matrix. A concise and informative derivation of this method is given in Ref. 7. Whereas Ref. 6 measured the size of the covariance matrix by its determinant, an alternative standard measure on the covariance matrix is its trace.[8] This measure has a direct interpretation as the sum of the variances of the estimated parameters. As shown herein, it is also closely related to the linear independence of the columns of the modal matrix through a condition number defined on the Frobenius norm. In this Note the backward elimination and the sequential replacement approaches are applied to the sensor placement problem using the covariance trace criterion. An extension to a condition number of the modal matrix is also given.

Method 1: Sensor Location to Minimize Variance

It is assumed that an approximate set of N_t target modes are available typically from a finite element analysis of the test structure. The mode shapes are defined at l locations on the structure. The response of the structure for the target modes may be written as

$$y(t) = \boldsymbol{\Phi}_l \eta(t) + w(t) \tag{1}$$

where $y(t)$ is an $l \times N_t$ vector of responses, and $\boldsymbol{\Phi}_l$ is an $l \times N_t$ modal matrix of target modes. The subscript notation for this matrix refers to the number of locations or rows. Here, $\eta(t)$ is the $N_t \times 1$ vector of modal responses (including the modal participation factors) and $w(t)$ is the $l \times 1$ vector of measurement noise. The correlation matrix of this random vector is

$$E[w(t)w(t-\tau)^T] = I\delta(t-\tau) \tag{2}$$

where E is the expectation operator, and δ is the Kronecker delta function. The best linear unbiased estimate of $\eta(t)$, which is simply the least squares estimate, is

$$\hat{\eta}(t) = \left[\boldsymbol{\Phi}_l^T \boldsymbol{\Phi}_l\right]^{-1} \boldsymbol{\Phi}_l^T y(t) \tag{3}$$

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The covariance matrix for $\hat{\eta}(t)$ is the inverse of the Fisher information matrix (FIM) F_l, i.e.

$$Q = F_l^{-1} = E[\hat{\eta}(t)\hat{\eta}^T(t)] - \eta(t)\eta^T(t) = \gamma^2\left[\Phi_l^T\Phi_l\right]^{-1} \tag{4}$$

The elements of the diagonals of Q are the variances of the individual modal responses. That is,

$$E\left[\hat{\eta}_i^2(t)\right] - \eta_i^2 = Q_{ii} \tag{5}$$

The total variance for all the modal responses, which is also the trace of the covariance matrix, is given by

$$\sigma^2 = \sum_{i=1}^{N_t} Q_{ii} \tag{6}$$

Noting that the Fisher information matrix may be written as

$$F_l = 1/\gamma^2 \sum_{i=1}^{l} \phi_i\phi_i^T \tag{7}$$

where ϕ_i is the ith column of Φ^T, the change in the FIM F_l, after deleting the ith sensor, is

$$F_{l-1} = F_l - \phi_i\phi_i^T \tag{8}$$

the new trace of the covariance matrix, using the matrix inversion lemma, is

$$\mathrm{tr}\left[F_{l-1}^{-1}\right] = \mathrm{tr}\left[F_l^{-1}\right] + \frac{\phi_i^T F_l^{-1} F_l^{-1}\phi_i}{1 - \phi_i^T F_l^{-1}\phi_i} = \mathrm{tr}\left[F_{l-1}^{-1}\right] + \Delta_i\sigma^2 \tag{9}$$

where the increase in variance due to deletion of the ith sensor is $\Delta_i\sigma^2$. The truncated singular value decomposition of Φ_l is defined as

$$\Phi_l = U_l\Lambda_l V_l^T \tag{10}$$

where U_l is the $l \times N_t$ matrix of left-orthonormal singular vectors, Λ_l is a diagonal matrix of singular values, and V_l is the $N_t \times N_t$ matrix of right-orthonormal singular vectors. Using the orthonormal properties of the singular vectors

$$F_l^{-1} = V_l\Lambda_l^{-2}V_l^T \tag{11}$$

From Eq. (10)

$$\phi_i^T V_l = u_i^T \Lambda_l \tag{12}$$

where u_i^T is the ith row of U_l, then $\Delta_i\sigma^2$ from Eq. (9) becomes

$$\Delta_i\sigma^2 = \frac{u_i^T\Lambda_l^{-2}u_i}{1 - u_i^T u_i} = \frac{\sum_{j=1}^{N_t} u_{ij}^2/\lambda_j^2}{1 - \sum_{j=1}^{N_t} u_{ij}^2} = \frac{[\Phi\Phi^T]_{ii}^{\dagger}}{[I - \Phi\Phi^{\dagger}]_{ii}} \tag{13}$$

The location giving the minimum value of $\Delta_i\sigma^2$ is deleted and the process repeated until reaching the required number of sensors. For a complete set of left-orthonormal singular vectors $\sum_{j=1}^{l} u_{ij}^2 = 1$ and since $l \geq N_t$, this quantity has to be less or equal to 1. Any sensor attaining this upper bound, during a stage of the sensor selection process, must not be deleted at that stage.

Method 2: Local Iteration

An alternative approach to the backward elimination scheme would be to compute a solution for the required number of sensors directly. This is known as a sequential replacement method and is implemented by refining an initial set of locations until an optimum solution is obtained. The variance after deleting the ith location and adding the jth location can be shown to be

$$\mathrm{tr}\left[\bar{F}_m^{-1}\right] = \mathrm{tr}\left[F_m^{-1}\right] + \frac{\phi_i^T F_m^{-1} F_m^{-1}\phi_i}{1 - \phi_i^T F_m^{-1}\phi_i} - \frac{\phi_j^T F_{m-1}^{-1} F_{m-1}^{-1}\phi_j}{1 + \phi_j^T F_{m-1}^{-1}\phi_j} \tag{14}$$

where $\bar{F}_m$ is the updated Fisher information matrix constructed using m locations, F_m is the original FIM before exchanging locations i and j, and F_{m-1} is obtained by deleting the ith location, i.e.,

$$F_{m-1} = F_m - \phi_i\phi_i^T \tag{15}$$

The proposed approach is to iteratively refine the initial solution by deleting the least effective location and adding a more effective location. A more effective location is one that decreases the original variance before the exchange of locations. The iterations are stopped when no decrease in the variance is possible. This method is guaranteed to converge to an improvement over the original set of locations or to the original set of locations. It may, therefore, converge to a poor solution when the initial set of locations are very far from optimum. The selection of the initial set of locations is, therefore, crucial.

Selection of Initial Locations for Method 2

A backward elimination strategy based on minimizing the correlation between the rows of the modal matrix is used to arrive at an initial set of locations. The algorithm consists of forming the correlation matrix between all of the normalized rows of the modal matrix and then taking the norm of each column of the correlation matrix. That is,

$$C = \bar{\Phi}\bar{\Phi}^T \tag{16}$$

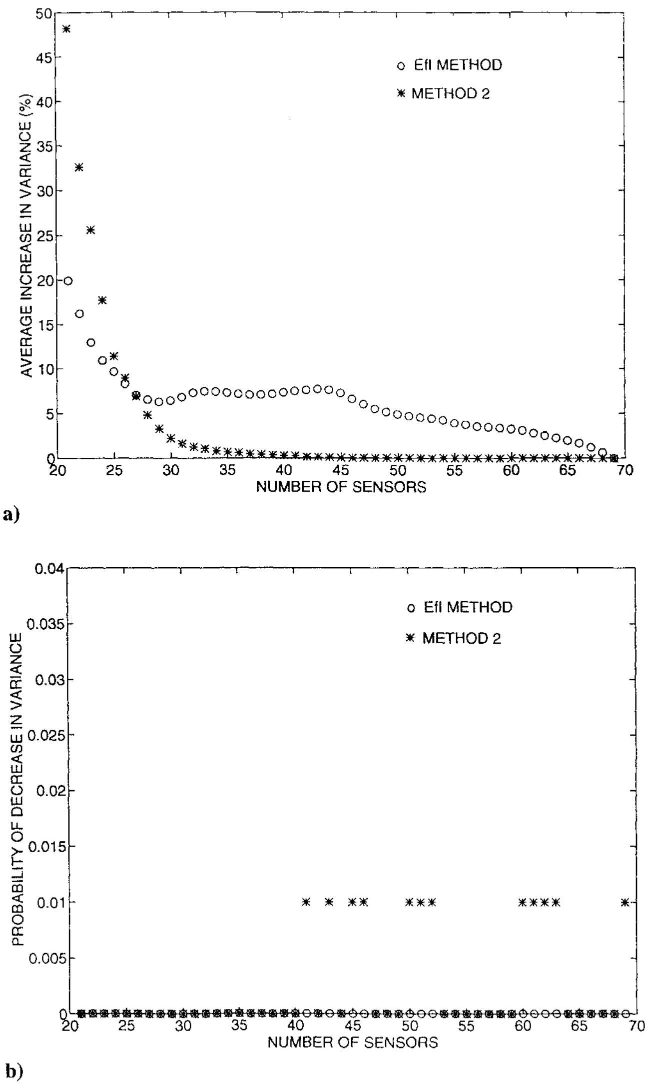

Fig. 1 Comparison of variance of EfI method and method 2 to method 1: a) percentage increase in variance from method 1 to EfI method and method 2 and b) probability of higher variance from method 1.

where $\bar{\Phi}$ is obtained by normalizing the rows of Φ to unit norm. An element of C, c_{ij} is the cosine of the angle between the ith row and the jth row of the modal matrix. If the norm of the ith row of C is large it implies that the ith location is correlated with a substantial number of other locations and may be safely deleted. Preference may be given to locations with a larger signal content by (inversely) weighting the correlation values by the size of the rows. From several simulations a measure β_i defined by

$$\beta_i = \sum_{j=1}^{l} c_{ij}^2 / \|\phi_i\| \tag{17}$$

produces satisfactory results. The location with the maximum value is deleted and the process repeated until reaching the required number of locations. The second method is much faster than method 1 since the correlation matrix is only computed once.

Minimization of Condition Number

The condition number is a reliable measure of the linear independence of a matrix. The trace measure given earlier is directly related to the condition number of the modal matrix based on the Frobenius norm. Using the Frobenius norm the condition number of the modal matrix is

$$\kappa_F(\Phi_l) = \|\Phi_l\|_F \|\Phi_l^\dagger\|_F = \left[\mathrm{tr}(F_l)\mathrm{tr}\left(F_l^{-1}\right)\right]^{\frac{1}{2}} = \left[\mathrm{tr}(F_l)\sigma^2\right]^{\frac{1}{2}} \tag{18}$$

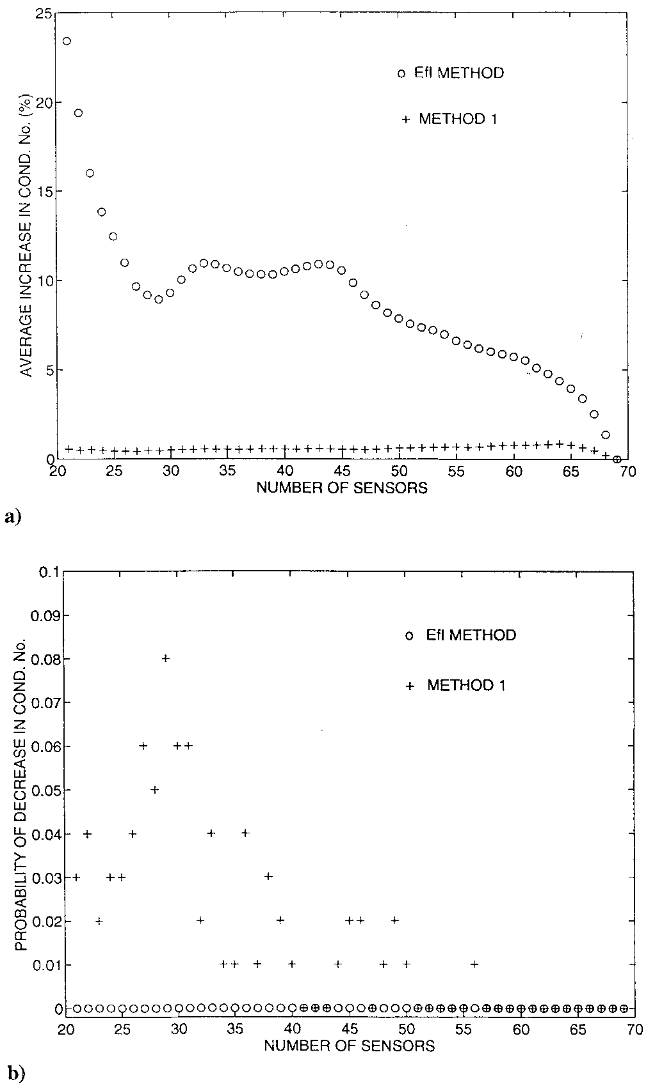

Fig. 2 Comparison of condition number technique to EfI method and method 1: a) percentage increase in condition number by using method 1 and EfI method and b) probability of higher condition number from condition number method.

where † denotes the pseudoinverse. Various error bounds dependent on κ_F as well as $\|\Phi^\dagger\|_F$ (i.e., σ^2) are given in Ref. 9. One may pick locations that minimize this condition number by computing

$$\alpha_i = \mathrm{tr}(F_l)\Delta_i\sigma^2 - \|\phi_i\|^2\left(\Delta_i\sigma^2 + \sigma^2\right) \tag{19}$$

The location giving the minimum value of α_i is deleted, σ^2 and $\mathrm{tr}(F_l)$ are easily updated using Eqs. (9) and (13), and the process is continued until reaching the required number of locations.

Results and Discussion

A series of simulations were performed to validate the expected behavior on the sensor location methods. Since the global minimum is not known, comparisons were performed with the EfI method which is known to produce good modal partitions. Using the finite element method 20 modes of an Euler–Bernoulli beam were calculated and interpolated at 400 evenly spaced locations along the span using beam shape functions. Since it is erroneous to draw conclusions about sensor location methods from a single simulation, 200 sensor location problems were generated by randomly selecting 80 candidate locations from the available parent set. Various numbers of locations from 21 to 79 were then chosen from each set of candidate locations. Figure 1a shows the average and minimum increase in variance in going from locations chosen by method 1 to those chosen by the EfI method and method 2. Figure 1b shows the probabilities of the EfI method and method 2 producing a lower variance than method 1. From Fig. 1, method 1 tends to minimize the sum of the variances of the estimated modal response as expected from its derivation, and method 2 produces locations of the same quality as method 1 until the number of sensors approaches the number of modes. It should, however, be noted that method 2 is much faster when the desired number of locations is much fewer than the candidate locations, a situation that occurs frequently in practice. The sequential replacement approach may also be easily applied to the EfI as well as the condition number criterion. Figure 1b shows that the conclusions of Fig. 1a occur with high probability.

Figure 2 shows the performance of the condition number method in comparison to both method 1 and the EfI method. It is seen that the method on the average produces a modal matrix with lower condition number than both the EfI method and method 1, and from Fig. 2b it does so with high probability.

Summary and Conclusions

Methods for placement of sensors for modal identification of structures were presented. Method 1 is an application of the backward elimination philosophy to a criterion based on the variance of parameters estimated using a modal matrix. The effectiveness of this method in minimizing the variance was confirmed using numerical simulations and a comparison with the EfI method.

A method for minimizing the condition number of the modal matrix partition was presented as an extension of method 1. Comparisons with the EfI method and method 1 using numerical simulations verified that the method generates modal partitions with improved condition numbers.

Method 2 is a local iteration method designed to save computation time compared to method 1. An approach for generating the initial set of locations was presented along with a method for refining the locations. This method was found to give results which are close to those of method 1 but with significantly less computation time. Methods 1 and 2 are directly significant within modal filtering where the end product is the estimate of the modal responses. All of the methods are applicable in general structural identification since the criteria considered preserve the linear independence of the columns of the modal matrix.

It should be noted that all of these methods are directly tied to modal filtering and only indirectly to other structural identification schemes such as model refinement, modal parameter estimation, force determination, etc. Comparisons based on such identification schemes could not, therefore, be based on any reasonable theoretical expectations. Optimal sensor location can be made more specific by targeting a particular estimation method, obtaining a scalar measure of the performance of that technique, and then minimizing that measure. An example is given in Ref. 10 for time domain modal parameter estimation. Such methods are significantly more complicated, and for many cases the solutions obtained using simpler methods such as just described, although not best for the given application, would be sufficient.

References

[1]DeLorenzo, M. L., "Sensor and Actuator Selection for Large Space Structure Control," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 2, 1990, pp. 249–257.

[2]Maghami, P. G., and Joshi, S. M., "Sensor/Actuator Placement for Flexible Space Structures," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 29, No. 2, 1993, pp. 345–351.

[3]Bayard, D. S., Hadaegh, F. Y., and Meldrum, D. R., "Optimal Experiment Design for Identification of Large Space Structures," *Automatica*, Vol. 24, No. 3, 1988, pp. 357–364.

[4]Salama, M., Rose, T., and Garba, J., "Optimal Placement of Excitations and Sensors for Verification of Large Dynamic Systems," *Proceedings of the AIAA/ASME/ASCE/AHS 28th Structures, Structural Dynamics, and Materials Conference*, AIAA, Washington, DC, 1987, pp. 1024–1031.

[5]Lim, T. W., "Sensor Placement for On-Orbit Modal Testing," *Journal of Spacecraft and Rockets*, Vol. 29, No. 2, 1992, pp. 239–246.

[6]Kammer, D. C., "Sensor Placement for On-Orbit Modal Identification and Correlation of Large Space Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 2, 1991, pp. 251–259.

[7]Kammer, D. C., "Reply by Author to W. L. Poston and R. H. Tolson," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 6, 1992, p. 1514.

[8]Fedorov, V. V., *Theory of Optimal Experiments*, Academic, New York, 1972, pp. 52–55.

[9]Stewart, G. W., "Stochastic Perturbation Theory," Dept. of Computer Science, Univ. of Maryland, Rept. UMIACS-TR-88-76, College Park, MD, Oct. 1988.

[10]Liu, C., and Tasker, F. A., " Sensor Placement Using Time Domain Perturbation Approach," *Proceedings of the AIAA Dynamics Specialists Conference* (Hilton Head, SC), AIAA, Washington, DC, 1994, pp. 163–172.

New Filtering Method for Linear Weakly Coupled Stochastic Systems

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I. Introduction

MODELS of many real dynamical control systems possess weakly coupled structures such as communication satellites,[1] helicopters,[2] and power systems.[3] The weakly coupled systems also represent the linearized models of dynamical systems described by partial differential equations in the modal coordinates.[4,5] In this Note we present a new method for optimal filtering of linear weakly coupled stochastic systems. The obtained results are applied to the filtering problem of a helicopter model obtained in Ref. 2.

The filtering problem of linear weakly coupled systems has been well documented in the control theory literature.[3,6,7] It is important to point out that the local filters in Refs. 3, 6, and 7 are driven by the innovation processes so that additional communication channels are required to form the innovation processes. In the newly proposed scheme these filters will be driven by the system measurements only. In addition, the optimal filter gains will be completely determined in terms of the exact reduced-order local algebraic Riccati equations.

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Consider the linear continuous-time invariant weakly coupled stochastic system

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + \varepsilon A_2x_2 + G_1w_1 + \varepsilon G_2w_2 \\ \dot{x}_2 &= \varepsilon A_3x_1 + A_4x_2 + \varepsilon G_3w_1 + G_4w_2 \end{aligned} \tag{1}$$

with the corresponding measurements

$$\begin{aligned} y_1 &= C_1x_1 + \varepsilon C_2x_2 + v_1 \\ y_2 &= \varepsilon C_3x_1 + C_4x_2 + v_2 \end{aligned} \tag{2}$$

where $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ are state vectors, $w_i \in R^{r_i}, i = 1, 2$, and $v_i \in R^{l_i}, i = 1, 2$, are zero-mean, stationary, white, Gaussian noise stochastic processes with intensities $W_i > 0$ and $V_i > 0$, respectively, and $y_i \in R^{l_i}$ are the system measurements. In the following $A_i, G_i, C_i, i = 1, 2, 3, 4$, are constant matrices.

II. New Method for Filter Decomposition

In the decomposition procedure from Refs. 3, 6, and 7, the local filters require additional communication channels necessary to form the innovation processes (see Fig. 1). Here, we propose a new decomposition scheme such that the local filters are completely decoupled and both of them are driven by the system measurements. The new method is based on the exact decomposition technique for solving the regulator algebraic Riccati equation of weakly coupled systems developed in Ref. 10. We give an additional interpretation of the results from Ref. 10, which will be used in this Note.

Consider the linear-quadratic optimal control problem corresponding to Eq. (1):

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + \varepsilon A_2x_2 + B_1u_1 + \varepsilon B_2u_2 \\ \dot{x}_2 &= \varepsilon A_3x_1 + A_4x_2 + \varepsilon B_3u_1 + B_4u_2 \\ J &= \int_0^\infty \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u^T R u \right] dt \\ & Q \ge 0, \qquad R > 0 \end{aligned} \tag{3}$$

where the control vector with components $u_i \in R^{m_i}, i = 1, 2$, has to be chosen such that the performance criterion J is minimized.

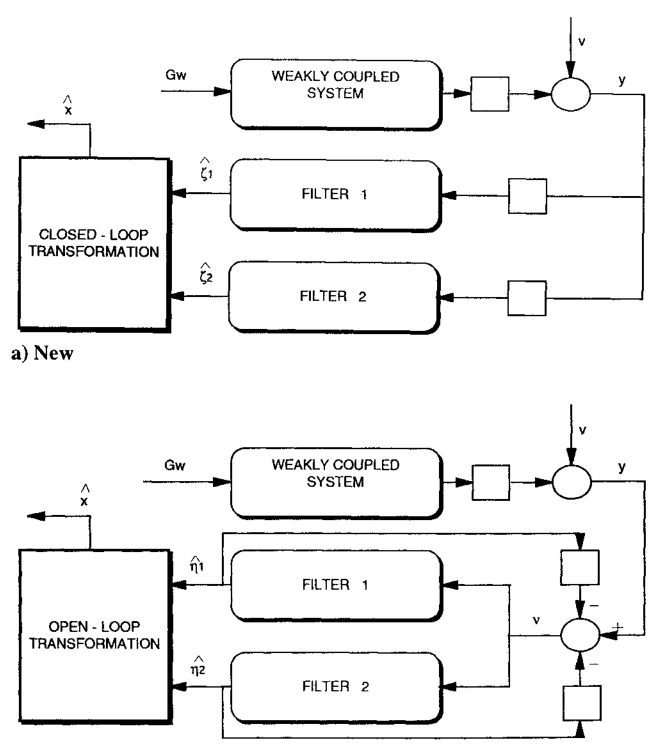

Fig. 1 Filtering methods.